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Heterotic strings with torsion

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Abstract: In this paper we describe the heterotic dual of the type-IIB theory compactified to four dimensions on a toroidal orientifold in the presence of fluxes. The type-IIB background is most easily described in terms of an $\mathcal{M}$-theory compactification on a fourfold. The heterotic dual is obtained by performing a series of U-dualities. We argue that these dualities preserve supersymmetry and that the supergravity description is valid after performing them. The heterotic string is compactified on a manifold that is no longer Kähler and has torsion. These manifolds have to satisfy a number of constraints, for example, existence of an holomorphic three-form, size limits, torsional equations etc. We give an explicit form of the background and study the constraints associated to them.

Keywords: Superstrings and Heterotic Strings, M-Theory, F-Theory, Supergravity Models.

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Contents

1. Introduction

The standard model has had great success in describing elementary particles and their interactions. But since the standard model contains many free parameters we hope that there exists a theory beyond the standard model capable of explaining quantities like the pattern of quark and lepton masses or the size of the gauge hierarchy. String theory is an excellent candidate to be such a theory since it does not contain any free dimensionless parameters at all. Instead it has many scalar fields, the moduli fields, that describe the different shapes and sizes of the internal manifold. The parameters of the standard model should then ultimately be fixed in terms of the vacuum expectation values of these fields. To leading order in the supergravity approximation string theory contains many degenerate ground states labelled by the moduli fields since their expectation values cannot be determined. This is an unattractive situation since it seriously limits the predictive power of string theory.

The string and $\mathcal{M}$-theory effective actions contain quantum gravity corrections which are of higher order in derivatives. After taking these into account the tensor fields acquire non-vanishing expectation values. The first example of this phenomenon were found in [1, 2] for compactifications of the heterotic string. In the presence of these fluxes the moduli fields are no longer arbitrary. Indeed, in [3] and [4] the constraints on the moduli fields for supersymmetric type-IIB compactifications to four dimensions were obtained by lifting the $\mathcal{M}$-theory compactifications on a Calabi-Yau 4-fold found in [5] to $\mathcal{F}$-theory.

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In the context of $\mathcal{M}$-theory compactifications on Calabi-Yau 4-folds the fluxes give rise to a supersymmetric background if

$$G \wedge J = 0 \qquad \text{and} \qquad G_{(3,1)} = G_{(4,0)} = 0\,, \tag{1.1}$$

where G is the 4-form of eleven-dimensional supergravity and J is the Kähler form of the internal manifold. The first equation in (1.1) determines the Kähler moduli and the second equation determines the complex structure.

The supersymmetric backgrounds are not the only solutions of the equations of motion. Indeed, any self-dual flux, i.e. any flux that satisfies $G = \star G$, will solve the equations of motion. So it is consistent to turn on a flux proportional to the holomorphic $(4,0)$-form of the Calabi-Yau or proportional to $J \wedge J$ [4, 6]. These additional vacua have two intriguing properties: they break supersymmetry and they have a vanishing cosmological constant. This is certainly very interesting from the phenomenological point of view.

If we include fluxes, besides having vacua with a vanishing cosmological constant and a broken supersymmetry, it is possible to determine the ground state of the theory, as we have seen. Moreover in compactifications with fluxes a non-vanishing warp factor has to be included in the metric. This warp factor provides one of the few known mechanisms for naturally generating a large hierarchies of physical scales. Recently it has been speculated that the masses generated by the fluxes could even be at the phenomenological viable TeV scale [7]. All these properties are essential in order to make contact between string theory and the real world.

In this paper we would like to understand if these properties can also be found in compactifications of the heterotic string.

In the context of the heterotic string compactified to four dimensions it is also possible to include fluxes. But the situation is a bit different than in type-II compactifications. Indeed, if we compactify the heterotic string on a Calabi-Yau 3-fold non-vanishing fluxes will always break supersymmetry. The difference to type-IIB compactifications arises because there is only one $\mathcal{H}$-field for the heterotic string rather than two. It is only possible to turn on a flux proportional to the $(3,0)$-form [8]. If we start with a manifold that is non-Kähler and has torsion a flux is needed rather than forbidden to have an unbroken supersymmetry [1, 2, 9]. The space-time metric will no longer be a direct product but we have to include a warp factor.

In this paper we discuss the background for a compactification of the heterotic string on a manifold with torsion. In section 2 we start by constructing examples of $\mathcal{M}$-theory compactifications with fluxes to three dimensions on $T^8/\mathcal{G}$ where $\mathcal{G}$ is an orbifold group. One of the examples is the $\mathcal{M}$-theory lift of a compactification on the orientifold $T^6/\mathbb{Z}_2$ recently discussed by Kachru, Schulz and Trivedi (KST) [10]. We discuss the massless multiples that are present in three dimensions and find agreement with the field content of the KST model in one example. In section 3 we lift the $\mathcal{M}$-theory compactification on $T^4/\mathcal{G} \times T^4/\mathcal{G}'$ to $\mathcal{F}$-theory or equivalently to a compactification of the type-IIB theory to four dimensions. By performing a series of U-duality transformations we find the background for the compactification of the type-I/heterotic string to four dimensions. We obtain the metric and fluxes for the type-I/heterotic side. In section 4 we consider a concrete example of an heterotic dual. The six-dimensional internal manifold we get is no longer Kähler and has torsion since on the type-IIB side we started with non-vanishing NS-NS fluxes. These manifolds were described by Hull [1], Strominger [2] and by de Wit-Smit-Hari Dass [9].[1] An explicit compact example was first given in [3]. In the later part of the paper we show in detail that the background satisfies the torsional constraints that are imposed by supersymmetry. We conclude with some comments on various extensions of our idea.

2. $\mathcal{M}$-theory compactified on $T^8/\mathcal{G}$

2.1 Type-IIB string theory on the orientifold $T^6/\mathbb{Z}_2$

Recently motivated by [3] Kachru-Schulz-Trivedi [10] and Frey-Polchinski [12] studied a novel type-IIB compactification on the $T^6/\mathbb{Z}_2$ orientifold in the presence of NS-NS and R-R 3-form fluxes. The orientifold action can be denoted by $\Omega(-1)^{F_L}\mathcal{I}_6$ where $\mathcal{I}_6$ stands for the reflection of all the compactified dimensions. In this compactification many of the moduli get fixed. In the presence of 3-form fluxes the superpotential

$$W = \int G_3 \wedge \Omega\,, \tag{2.1}$$

is generated. Here Ω is the unique holomorphic $(3,0)$-form on T^6, and G_3 is given in terms of the NS-NS 3-form H and the R-R 3-form H' by $G_3 = H' - \varphi H$. We denote the axion-dilaton combination by $\varphi = \tilde{\phi} + ie^{-\phi}$. Observe that W survives on $T^6/\mathbb{Z}_2$.

The supersymmetry preserving background follows from minimizing the superpotential with respect to the dilaton and the complex structure τ_{ij} of the torus

$$W = 0\,, \qquad \partial_\phi W = 0 \qquad \text{and} \qquad \partial_{\tau_{ij}} W = 0\,. \tag{2.2}$$

Solving (2.2) KST showed explicitly how many of the moduli in this problem pick up some definite values.

The compactification of the type-IIB theory on $T^6/\mathbb{Z}_2$ can also be studied directly from the $\mathcal{M}$-theory point of view where many of the equations get simplified. In fact, the type-IIB model follows from a simple orbifold of $\mathcal{M}$-theory in the presence of G-fluxes. In the next couple of sections we shall elaborate this. We will show how the multiplet counting, anomaly analysis etc. result from the $\mathcal{M}$-theory point of view.

2.2 $\mathcal{M}$-theory lift

Let us now consider the $\mathcal{M}$-theory lift of the type-IIB model in some detail. The base of the 4-fold is $T^6/\mathbb{Z}_2$ where, as we have seen above, $\mathbb{Z}_2$ is an orientifold action. The 4-fold should be a torus fibration over the base. So we have two possibilities:

[1]The torsional backgrounds have also been addressed in a series of papers by Gates et al. [11]. However they considered a torsion that is closed. This is not the case we would like to consider here.

1. Over each fixed point of $T^6/\mathbb{Z}_2$ the fiber degenerates as $T^2/\mathcal{I}_2$ and everywhere else the fiber is T^2. This would correspond to the 4-fold $T^8/\mathcal{I}_8$ where $\mathcal{I}_n$ is a pure orbifold action acting on the coordinates x_i of T^n as $x_i \to -x_i$ for $i = 1, \ldots, n$.

2. We could also have the orientifold action of the base to be $\mathbb{Z}_2 = \Omega(-1)^{F_L}\mathcal{I}_2 \times \mathcal{I}_4$ so that the 4-fold becomes $T^8/\mathcal{G} \equiv T^4/\mathcal{I}_4 \times T^4/\mathcal{I}_4$.

For both cases the Euler characteristic can be related to the perturbative spectrum of massless multiplets. In the absence of any fluxes we expect the Euler characteristic to be given by

$$\chi = 24n\,, \tag{2.3}$$

where n is the number of M2-branes.

From the ten-dimensional point of view this would correspond to the type-IIA string theory on $T^8/\mathcal{G}$, where $\mathcal{G}$ can be any of the above two actions. Under a single T-duality of one of the cycles the $\mathcal{G}$ action goes to $\mathcal{G}' = (-1)^{F_L}\mathcal{G}$ and we have the type-IIB theory on $T^8/\mathcal{G}'$. Here we have n elementary IIB strings which are required to cancel the 2-form NS sector tadpole [13, 14].

The Euler characteristic can now be given in terms of n_1 (massless states in the NS-R sector) and n_2 (massless states in the R-R sector) as [14]

$$\chi = 2n_1 - n_2\,, \tag{2.4}$$

where we have used $\int X_8 = -\chi/24$. Here X_8 is a polynomial of fourth order in the Riemann tensor whose definition can be found for example in [15]. The values of n_1 and n_2 can be argued from the massless spectra in type IIB.

For $T^8/\mathcal{I}_8$ we have

$$n_1 = -64 \qquad \text{and} \qquad n_2 = 256\,. \tag{2.5}$$

The negative sign for n_1 can be explained as follows. The state surviving the GSO projection from the left, as well as the $\mathcal{I}_8$ projection is $(\mathbf{8_v})_L \times (\mathbf{8_s})_R$. Here L, R denote the left and right moving sectors of the type-II string while $\mathbf{8_{v,s}}$ are the SO(8) vector and spinor representations. Since $(\mathbf{8_s})_R$ has $(-1)^{F_R} = -1$ these contribute a net factor of -64.

For $T^8/\mathcal{G} \equiv T^4/\mathcal{I}_4 \times T^4/\mathcal{I}_4$ we have

$$n_1 = -160 \qquad \text{and} \qquad n_2 = 256\,. \tag{2.6}$$

There is a small subtlety that we mention at this point. For the 4-fold $T^8/\mathcal{I}_8$ the R-R sector twisted states all have $(-1)^{F_R} = 1$, however the NS-R twisted sectors are removed because there are no massless states.

Let us now determine the massless multiplets from compactifying the type-IIA theory (or $\mathcal{M}$-theory) on the above 4-folds. Let h_{ij} be the Hodge numbers of the 4-fold. As we know, the metric g_{MN} contributes $h_{11} + 2h_{31}$ non-chiral scalars, the tensor fields B_{MN} and C_{MNP} contribute h_{11} and $2h_{21}$ non-chiral scalars respectively. The spin-3/2 particles come from the gravitinos and their number is given by the Dirac index of the 4-fold. These ten-dimensional gravitinos also contribute to the spin-1/2 particles, and their number is given by the Rarita-Schwinger index. These indices are given by

$$\mathrm{ind}(\not{D}_{3/2}) = 4N - \frac{1}{6}\Big(p_2 - \frac{p_1^2}{4}\Big) \qquad \text{and} \qquad \mathrm{ind}(\not{D}_{1/2}) = \frac{N}{2}\,, \tag{2.7}$$

where N is used to label the supersymmetry as $(N/2, N/2)$ and p_i are the Pontryagin classes. Therefore we have the following multiplets in two dimensions [16]

$$\Big(g_{\mu\nu}, \phi, \frac{N}{2}\psi_\mu, \frac{N}{2}\psi\Big) \oplus \frac{2k}{N}\Big(\frac{N}{2}\phi, \frac{N}{2}\psi\Big)\,, \tag{2.8}$$

where the first one is the gravity multiplet and the second one is the non-chiral matter multiplet.

For the 4-fold k is defined as

$$\frac{k}{2} = h_{11} + h_{21} + h_{31} \tag{2.9}$$

If the 4-fold is a Calabi-Yau manifold then the above expression for k can be further simplified to

$$k = \frac{\chi}{3} - 16 + 4h_{21}\,. \tag{2.10}$$

This is however not the complete spectrum. From (2.3) and (2.4) we see that the consistent vacuum also contains wandering fundamental strings. The collective coordinates of these fundamental strings contribute 8 copies of (ϕ, ψ) to the total number of $(N/2, N/2)$ non-chiral matter multiplets. Therefore a consistent vacuum will have m scalar multiplets where m is given by

$$m = \frac{2k}{N} + \frac{2}{3}\frac{|2n_1 - n_2|}{N}\,. \tag{2.11}$$

For $T^8/\mathcal{I}_8$ we have $N = 16$ and the Hodge numbers are $h_{11} = 16, h_{21} = 0, h_{31} = 16$ and $k = 64$. Therefore we have

$$(g_{\mu\nu}, \phi, 8\psi_\mu, 8\psi) \oplus 24(8\phi, 8\psi)\,, \tag{2.12}$$

massless multiplets in two dimensions. Observe that if we ignore the contributions from wandering strings then we find agreement with the multiplets of the type-IIB model considered in [10]. Indeed, if we compactify the examples considered in [10] on T^2, so that we would have the type-IIB theory compactified on $T^6/\mathbb{Z}_2 \times T^2$, then we would have a total of 65 scalars. This agrees exactly with the number of scalars found in (2.12) if we ignore the wandering strings.

For the $T^4/\mathcal{I}_4 \times T^4/\mathcal{I}_4$ case we have $N = 8$ and the Hodge numbers are $h_{11} = 40, h_{21} = 0, h_{31} = 40$ and $k = 160$. Therefore we have

$$(g_{\mu\nu}, \phi, 4\psi_\mu, 4\psi) \oplus 88(4\phi, 4\psi)\,, \tag{2.13}$$

massless non-chiral multiplets in two dimensions.

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2.3 Flux analysis and M2-branes

Let us now consider the situation in the presence of fluxes. $\mathcal{M}$-theory on a 4-fold $\mathcal{Y}$ can be related to IIB in two different ways. We can have $\mathcal{Y} \times T^2$ and shrink the two torus to zero size. This way we have type IIB on a 4-fold $\mathcal{Y}$. On the other hand if $\mathcal{Y}$ itself is a T^2 fibration over a base $\mathcal{B}$ then we get type IIB on a six-manifold $\mathcal{B}$.

For the case that we consider the type-IIB theory compactified on a 4-fold $\mathcal{Y}$ the two-dimensional vacuum preserves chirality and hence induces a vacuum momentum given by [17, 16]

$$L_0 - \bar{L}_0 = \frac{\chi}{24} \,. \tag{2.14}$$

This is computed by using the fact that a free periodic boson has vacuum energy $-1/24$ while a free periodic fermion has energy $1/24$. In the absence of fluxes this vacuum momentum is related to the O-plane and D-branes charges of the type-IIB theory compactified on $\mathcal{B}$.

For $\mathcal{Y} = T^8/\mathcal{I}_8$ and $\mathcal{B} = T^6/\mathbb{Z}_2$ the 4-form of eleven-dimensional supergravity G will satisfy

$$d \star G = -\frac{1}{2} G \wedge G - 4\pi^2 X_8 - 4\pi^2 \sum_{i=1}^{n} \delta^8(x - x_i) \,, \tag{2.15}$$

when $\mathcal{M}$-theory is compactified on the 4-fold. The Hodge $\star$-operator is defined with respect to the warped metric. We denote the number of M2-branes by n and are also using the membrane tension $T_3 = 1$.

The above equation integrates to give the anomaly condition. Therefore when the type IIB is compactified on the six-manifold the X_8 term should be related to the number of O3-planes and n will be correspond to the number of D3-branes. The $G \wedge G$ term will give rise to the $H \wedge H'$ term in type IIB. Therefore (2.15) will eventually become

$$16 = n - \int_{\mathcal{B}} H \wedge H' \,. \tag{2.16}$$

For the case when $\mathcal{Y} = K3 \times K3$ and $\mathcal{B} = T^2/\mathbb{Z}_2 \times K3$ the analysis is a little more elaborate. Going from $\mathcal{M}$-theory to type IIB gives us not only O-planes but also D-branes. In fact, as discussed earlier in many papers, we get 4 O7-planes and 16 D7-branes. These D7-branes when wrapped on the other K3 give rise to anti-D3-branes. Details of this have appeared in [18]. We would like to concentrate on the origin of the gauge fluxes on the D7-branes. The $\mathcal{M}$-theory origin of these fluxes are the *localized* G-fluxes near the singularities of $T^4/\mathcal{I}_4$. To make this a little more precise, let us concentrate on the region near one orbifold singularity. This local region will look like a Taub-NUT space. Therefore $\mathcal{M}$-theory is compactified on $TN \times K3$ where the various directions are

$$\begin{array}{lcccccccccc} TN: & - & - & - & - & - & - & 7 & 8 & 9 & 10 \\ K3: & - & - & 3 & 4 & 5 & 6 & - & - & - & - \end{array} \tag{2.17}$$

If we denote the metric of $TN \times K3$ by $g_{a\bar{b}}$, where $a, \bar{b}$ label the complex coordinates, and the metric of the flat space-time by $\eta_{\mu\nu}$, then the metric for the whole system will be warped as

$$ds^2 = e^{-\Phi(y)} \eta_{\mu\nu} dx^\mu dx^\nu + e^{\frac{1}{2}\Phi(y)} g_{a\bar{b}} dy^a dy^{\bar{b}} \,, \tag{2.18}$$

where $\Phi(y)$ is the warp factor.

The G-flux can be expanded as[2]

$$\frac{G}{2\pi} = dz \wedge \omega_1 + d\bar{z} \wedge \omega_2 + \sum_{i=1}^{4} F^i \wedge \Omega^i \,, \tag{2.19}$$

where $i = 1, \ldots, 4$ labels the four set of singularities on the base $T^2/\mathcal{I}_2$ of $T^4/\mathcal{I}_4$, z is the complex coordinate of the fibre and ω is a 3-form on the base $\mathcal{B} = T^2/\mathbb{Z}_2 \times K3$. As we know, at every fixed points of the base the fiber is $T^2/\mathcal{I}_2$ with four fixed points. Therefore one set of singularities in our notation will have four fixed points. Ω^i are the harmonic two-forms on $T^4/\mathcal{I}_4$. The zero-modes of the two-forms satisfy the condition

$$\nabla_{[m} \Omega^i_{np]} = 0 \,. \tag{2.20}$$

The solution of (2.20) can be divided into self-dual and anti-self-dual parts. The self-dual part, which is a singlet of the holonomy group SU(2), is covariantly constant tensor and is not normalisable. We choose the anti-self-dual part which is normalisable and hence localized at the fixed points of $T^4/\mathcal{I}_4$. These localized forms are used to get gauge fields $F^i_{a\bar{b}}$ on the D7-brane world-volume. The X_8 term of $\mathcal{M}$-theory now contributes to the gravitational couplings on both D7-branes and O7-planes. The warp factor satisfies

$$\begin{aligned} \tilde{\star}\nabla^2 e^{3\Phi/2} &= H \wedge H' + (4\pi^2\alpha')^2 \sum_{i=1}^{n} \delta^{(6)}(x - x_i) + \\ &\quad + \frac{(\alpha'\pi)^2}{4} \sum_{j=1}^{4} \left[\mathrm{tr}(R \wedge R) - \mathrm{tr}(F^j \wedge F^j) \right] \delta^{(2)}(x - x_j) \,, \end{aligned} \tag{2.21}$$

where n is the number of D3-branes and the sum is over four copies of one O7-plane plus four D7-branes. The factor of a quarter in front of the curvature terms can be calculated by carefully taking the gravitational couplings of D7- and O7-planes into account [22]. The Hodge $\tilde{\star}$ and the laplacian are with respect to the unwarped metric.

For $\mathcal{M}$-theory on $T^8/\mathcal{I}_8$ one can show that there are *no* normalisable harmonic 2-forms at the orbifold singularities. This is related to the fact that $T^8/\mathcal{I}_8$ cannot be blown up to a smooth Calabi-Yau. In other words the blow up operators are irrelevant rather than marginal. This is consistent because we only have O3-planes from the singularities and no D-branes.[3]

[2]This equation has also appeared in [19] and [20] in a different context. In [19] this is used to understand the supergravity dual of $\mathcal{N} = 1$ theories and in [20], inflationary scenario. It is shown in [20] that choosing ω_i to be non-primitive, we could still make G primitive and this could be used to reach a supersymmetric background which is the minima of the hybrid inflationary-potential [21].

[3]An alternative way to see this is as follows. To get gauge fluxes we need $h^{1,1} = 256$ but from Hodge diamond $T^8/\mathcal{I}_8$ has $h^{1,1} = 16$. All these are from the untwisted states and therefore constant.

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Finally let us understand the $\mathcal{M}$-theory lift of the exotic O3-planes discussed in [10]. We shall be brief here and the reader wishing to understand more should consult the literature [23, 24, 25, 12, 26]. In the above discussion there are four 2-planes each with charges $-1/16$. Combining them we get a total charge of $-1/4$ which is the expected value of a $O3$-plane in IIB. To get the exotic $O3$-plane one has to combine two 2-planes and two $OM2^+$ planes. These $OM2^+$ planes have charges $-3/16$ each. Combining them we get the exotic $O3$ charge of $1/4$.

3. A compactification of the heterotic string on a manifold with torsion

In this section we will study a specific example of Heterotic SO(32) string compactification with torsion. We begin with type-IIB theory compactified on an orientifold and perform a series of U-duality transformations to reach our heterotic model. The type-IIB theory that we start off with is compactified on $T^6/\mathbb{Z}_2$. The $\mathbb{Z}_2$ action is $\Omega(-1)^{F_L}\mathcal{I}_6$ and $\mathcal{I}_6 = \mathcal{I}_{897654}$ reverses the all the tori directions. In the following note we shall however assume that the $\mathbb{Z}_2$ action acts as $T^2/\mathbb{Z}_2 \times T^4/\mathcal{I}_4$ where $\mathbb{Z}_2 = \Omega(-1)^{F_L}\mathcal{I}_{89}$ and $\mathcal{I}_4 = I_{7654}$.

We also have a nontrivial H and H' background in type IIB. However we should be careful about the choice of the H and H' fields. It is known that under the orientifold action $\Omega(-1)^{F_L}$ the two H fields are transformed by the $\mathrm{SL}(2,\mathbb{Z})$ matrix

$$M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (3.1)$$

Therefore only those fields which have one of their components along the torus directions will survive this projection. Thus our H and H' backgrounds have a leg along the T^2 parametrized by the $8, 9$ coordinates.

The two T-dualities along the two circles of $T^2/\mathbb{Z}_2$ will convert $\Omega(-1)^{F_L}\mathcal{I}_{89}$ to Ω [18]. In other words it will take our orientifold theory to type-I theory. In what follows we will specify the exact backgrounds for the type-I theory. In the special case when we start with type IIB with $H = H' = 0$, the type-I theory that we get by performing two T-dualities is compactified on $K3 \times T^2$. In the presence of backgrounds this is no longer the case. On the type-I side we get a six-dimensional manifold which is a non-Kähler complex manifold with vanishing first Chern class [3]. And there is a nontrivial antisymmetric field background on it which we can specify exactly.

But before we go into this we should first see how the couplings and volumes of the compactified space change under the series of U-duality transformations.

3.1 The coupling constant maps

Let us take the type-IIB theory compactified on $T^2/\mathbb{Z}_2 \times T^4/\mathcal{I}_4$. We take the volumes of $T^2/\mathbb{Z}_2$ and $T^4/\mathcal{I}_4$ to be $\tilde{v}$ and v respectively. In total the six-dimensional internal manifold has a volume $V = v\tilde{v}$. We also define $\tilde{v} = R_1 R_2$ where R_1, R_2 are the two radii of $T^2/\mathbb{Z}_2$.

Under two T-duality transformations, which take $R_i \to \alpha'/R_i$ $(i = 1, 2)$, we go from the type-IIB theory to the type-I theory. The various couplings constants and volumes can

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now be specified in terms of type-IIB variables:

$$g_I = \frac{\alpha' g_B}{\tilde{v}}\,, \qquad \tilde{v}_I = \frac{\alpha'^2}{\tilde{v}}\,, \qquad v_I = v \qquad \text{and} \qquad g_I^{(4)} = \frac{g_B}{\sqrt{v\tilde{v}}}\,. \tag{3.2}$$

Here g_I and g_B denote the ten-dimensional type-I and type-IIB couplings respectively. By $g_I^{(4)}$ we denote the four-dimensional type-I coupling.

Under an S-duality transformation we get the heterotic SO(32) theory. The various couplings and volumes in this theory can again be written in terms of type-IIB variables:

$$g_{\rm het} = \frac{\tilde{v}}{\alpha' g_B}\,, \qquad \tilde{v}_{\rm het} = \frac{\alpha'}{g_B}\,, \qquad v_{\rm het} = \frac{v\tilde{v}^2}{\alpha'^2 g_B^2} \qquad \text{and} \qquad g_{\rm het}^{(4)} = \sqrt{\frac{g_B}{v\alpha'}}\,. \tag{3.3}$$

In order to get (3.3) we have used the SO(32) heterotic-type-I relations [27]

$$\begin{aligned} G^{\rm het}_{MN} &= g_I^{-1} g^I_{MN}\,,\\ g_{\rm het} &= g_I^{-1}\,. \end{aligned} \tag{3.4}$$

Observe that the volume $\tilde{v}_{\rm het}$ is independent of $\tilde{v}$ and only depends on the type-IIB coupling.

In compactifications of the type-IIB theory to four dimensions with fluxes the dilaton is generically stabilized [28]. Therefore the type-IIB coupling is $g_B \approx 1$. Let us consider the large volume limit of the type-IIB theory:

$$g_B \approx 1\,, \qquad \tilde{v} \gg 1 \qquad \text{and} \qquad v \gg 1\,. \tag{3.5}$$

From our mapping to type I we find

$$g_I^{(4)} \to 0 \qquad \text{and} \qquad V_I = \frac{\alpha'^2 v}{\tilde{v}}\,. \tag{3.6}$$

Therefore the six-dimensional compact type-I volume can be adjusted by picking some definite ratio between the two volumes in type-IIB keeping their individual values large.

For the SO(32) heterotic case we find:

$$g_{\rm het}^{(4)} \to 0 \qquad \text{and} \qquad V_{\rm het} \gg 1\,. \tag{3.7}$$

From the above analysis we see that we can use supergravity analysis to study the various four-dimensional theories. There is a subtlety related to the large volume limits considered here. We shall comment on this in later section.

3.2 Type-IIB orientifold background

To discuss the type-IIB orientifold background the starting point is the effective action. In the absence of sources it is given by,[4] [27]

$$S_{\rm IIB} = S_{\rm NS} + S_{\rm RR} + S_{\rm CS}\,, \tag{3.8}$$

[4]Throughout the paper we shall always be in the string-frame, except in section 4.2 where we will go briefly to Einstein-frame. It will hopefully be clear from the text when we switch frames. Also in our notations $\tilde{g}_{mn}$ will be unwarped metric of six compact directions. Type-IIB coupling constant will be denoted by g_B and we shall keep it throughout the text even though we take examples where $g_B = 1$. Moreover the Hodge $\star$ will always signify duality wrt warped metric, and $\tilde{\star}$ wrt unwarped metric – unless mentioned otherwise.

with

$$\begin{aligned}
S_{\rm NS} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}e^{-2\phi}\left(R+4(\partial\phi)^2-\frac{1}{2\cdot 3!}H^2\right),\\
S_{\rm RR} &= -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-g}\left(F_1^2+\frac{1}{3!}\tilde{H}_3^2+\frac{1}{2\cdot 5!}\tilde{F}_5^2\right),\\
S_{\rm CS} &= -\frac{1}{4\kappa_{10}^2}\int D^+\wedge H\wedge H'\,.
\end{aligned} \tag{3.9}$$

The NS-NS and R-R field strengths H and H' are related to the potentials by

$$H = dB \qquad \text{and} \qquad H' = dB'\,. \tag{3.10}$$

The fields $\tilde{H}_3$ and $\tilde{F}_5$ are defined by

$$\begin{aligned}
\tilde{H}_3 &= H' - \tilde{\phi}H,\\
\tilde{F}_5 &= dD^+ - \frac{1}{2}B'\wedge H + \frac{1}{2}B\wedge H'\,.
\end{aligned} \tag{3.11}$$

The field strength F_1 is given by the derivative of the axion $F_1 = d\tilde{\phi}$.

The self-duality condition $\tilde{F}_5 = \star\tilde{F}_5$ cannot be derived from any known lagrangian and must be imposed by hand. In the absence of sources the equation of motion for $\tilde{F}_5$ is

$$d\tilde{F}_5 = H\wedge H'\,. \tag{3.12}$$

We will have D-branes and O-planes in the type-IIB bulk. As discussed in section 2.3 in the present example we will have 4 O7-planes and 16 D7-branes, that induce a negative D3-brane charge. These will introduce additional contributions to the effective action and equations of motion. The equation of motion for $\tilde{F}_5$ takes the form [22]

$$d\tilde{F}_5 = H\wedge H' + \rho, \tag{3.13}$$

where

$$\rho = (4\pi^2\alpha')^2\sum_{i=1}^{n}\delta^{(6)}(x-x_i) + \frac{(\alpha'\pi)^2}{4}\sum_{j=1}^{4}\left[\mathrm{tr}(R\wedge R)_j - \mathrm{tr}(F^j\wedge F^j)\right]\delta^{(2)}(x-x_j)\,, \tag{3.14}$$

In the presence of H and H' backgrounds and sources the metric in the string frame will have the following form:

$$ds^2 = \Delta(y)^{-1}\eta_{\mu\nu}dx^\mu dx^\nu + \Delta(y)\tilde{g}_{mn}dy^m dy^n\,, \tag{3.15}$$

where $\Delta(y)$ is the warp factor. The warp factor and 4-form potential are related by [29]

$$D_{\mu\nu\rho\lambda} = \frac{1}{g_B}\epsilon_{\mu\nu\rho\lambda}\Delta(y)^{-2}\,. \tag{3.16}$$

Therefore the equation of motion for $\tilde{F}_5$ is the determining equation for the warp factor.

3.3 T-dual map and type-I background

Let us start with our type-IIB orientifold model where the orientifold action has been described earlier. When we make two T-dualities along the two circles of $T^2/\mathbb{Z}_2$ the operator $\Omega(-1)^{F_L}\mathcal{I}_{89}$ changes to Ω and we get the type-I theory. We will assume all the fields and the warp factor to be independent of the torus directions. The H and H' fields are non-zero only on the internal space (i.e. on $T^2/\mathbb{Z}_2 \times K3$). Let us begin with the situation where the torus T^2 is not a square torus, i.e. $g_{89} \neq 0$.[5] Now we go to the type-I theory by T-dualising[6] first along the x^8 direction and then along the x^9 direction of the T^2.

In the next section we shall consider a particular orbifold example where the complex structures τ_{ij} of the tori involved are $i\delta_{ij}$. For such cases the metric of $T^2/\mathbb{Z}_2$ can be written in terms of "flat" coordinates w in the following way:

$$ds^2 = C\ dw d\bar{w}\,, \tag{3.17}$$

where C is a constant and w is defined as

$$dw = \prod_{i=1}^{4}\ (z - z_i)^{-\frac{1}{2}} dz\,. \tag{3.18}$$

However this does not mean that the metric is flat everywhere. At the four points z_i (which are the fixed points of $T^2/\mathbb{Z}_2$) there is a deficit angle of π. And of course, there is an overall warp factor. Using the earlier notation of Δ as the warp factor our metric is

$$\begin{pmatrix} \Delta^{-1}\eta_{\mu\nu} & 0 \\ 0 & \Delta\tilde{g}_{mn} \end{pmatrix}, \tag{3.19}$$

where $\tilde{g}_{mn}$ is the unwarped metric of $T^2/\mathbb{Z}_2 \times K3$.

With the above considerations we can now use Buscher's T-duality rules [30, 31, 32] to predict the exact form of type-I metric and the antisymmetric field background.[7] Since we are taking our torus to be non-diagonal the T-dual fields would be complicated. Let us denote the T-dual metric as g^I_{mn} where $m, n = 4, \ldots, 7$ labels the $K3$ directions and $m, n = 8, 9$ labels the tori directions. To simplify our formulae let us define two 2×2 matrices as

$$g = \begin{pmatrix} \tilde{g}_{88} & \tilde{g}_{89} \\ \tilde{g}_{89} & \tilde{g}_{99} \end{pmatrix} \qquad \text{and} \qquad g^I = \begin{pmatrix} g^I_{88} & g^I_{89} \\ g^I_{89} & g^I_{99} \end{pmatrix}, \tag{3.20}$$

where $\tilde{g}$ is as usual the unwarped metric. The metric in type-I theory is now related to the metric in IIB by the following simple relations:

$$\operatorname{tr} g^I = \frac{1}{\Delta} \operatorname{tr} g^{-1} \qquad \text{and} \qquad \det g^I = \frac{1}{\Delta^2} \det g^{-1}\,. \tag{3.21}$$

[5] Recall that the directions of $T^2/\mathbb{Z}_2 \times K3$ are $x^{9,8,\ldots,4}$ and $x^{0,1,2,3}$ are the usual coordinates of Minkowski space.

[6] We will assume $4\pi^2\alpha' = 1$ henceforth.

[7] In what follows we use the conventions of [32]. In this convention a p-form field will be defined as $(dA)_{\mu_1\mu_2\ldots\mu_{p+1}} = (p+1)\partial_{[\mu_1}A_{\mu_2\ldots\mu_{p+1}]}$ which symbolically can be denoted as $(dA)_{p+1} = (p+1)\partial A_p$. The convention of [31] is $(dA)_{p+1} = \partial A_p$. In differential geometry language a p-form will be defined with $1/p!$ i.e. $H = \frac{1}{p!} H_{\mu_1\ldots\mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$.

To find the g^I_{mn} let us define another 2×2 matrix:

$$b \equiv b_{(mn)} = \begin{pmatrix} B_{8m} & B_{8n} \\ B_{9m} & B_{9n} \end{pmatrix}. \tag{3.22}$$

On the $K3$ base these B-fields will form a vector bundle with globally defined field strength. Using (3.22) one can simplify considerably the other components of the type-I metric to:

$$g^I_{mn} = \Delta \tilde{g}_{mn} + \frac{\mathrm{tr}(b\sigma_4)\det(b\sigma_1 + g\sigma_3) + \mathrm{tr}(b\sigma_3)\det(b\sigma_2 + g\sigma_1)}{\Delta \det g}, \tag{3.23}$$

where σ_i are the Chan-Paton matrices

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.24}$$

The metric has also picked up off diagonal components which are related to the background B field by

$$g^I_{8m} = \frac{\det(b\sigma_1 + g\sigma_3)}{\Delta \det g} \qquad \text{and} \qquad g^I_{9m} = \frac{\det(b\sigma_2 + g\sigma_1)}{\Delta \det g}. \tag{3.25}$$

Analyzing the above equations we find that we no longer expect a simple product metric in type I. The metric is not only warped but also loses its simple product form. For the B^I fields we see that it depends on all the type-IIB fields like the 4-form field D_{mnpq}, the axion $\tilde{\phi}$ and the two B-fields:

$$\begin{aligned} B^I_{mn} &= D^+_{89mn} + 6B_{[9m}B'_{n8]} - 4\frac{\tilde{g}_{89}}{\tilde{g}_{88}}B_{8[m}B'_{n]8} - 2\tilde{\phi}B_{9[m}B_{n]8}, \\ B^I_{8m} &= -B'_{9m} + \tilde{\phi}B_{9m}, \qquad B^I_{9m} = B'_{8m} - \tilde{\phi}B_{8m}, \qquad B^I_{89} = -\tilde{\phi}. \end{aligned} \tag{3.26}$$

The above is therefore the complete background when we do not assume any approximations. Observe that the metric and the B fields depend non-trivially on the complex structure. In the next section we will however consider a slightly simplified background where

$$\tilde{g}_{89} = 0, \qquad \tilde{\phi} = 0 \qquad \text{and} \qquad D^+_{89mn} = 0. \tag{3.27}$$

The last equation is trivially true because the 4-form is not self-dual in the presence of H and H'. The set of equations (3.21), (3.23), (3.25) and (3.26) simplify drastically by applying (3.27):

$$g^I_{mn} = \Delta\tilde{g}_{mn} + \frac{B_{8m}B_{8n}}{\Delta\tilde{g}_{88}} + \frac{B_{9m}B_{9n}}{\Delta\tilde{g}_{99}}, \qquad g^I_{88} = \frac{1}{\Delta\tilde{g}_{88}}, \qquad g^I_{99} = \frac{1}{\Delta\tilde{g}_{99}}. \tag{3.28}$$

The off-diagonal components of the metric also take a very simple form. They depend only on the B fields:

$$g^I_{8m} = \frac{B_{8m}}{\Delta\tilde{g}_{88}}, \qquad g^I_{9m} = \frac{B_{9m}}{\Delta\tilde{g}_{99}}, \qquad g^I_{89} = 0, \tag{3.29}$$

together with the B^I fields reducing to

$$B^I_{mn} = 6B_{[9m}B'_{n8]}, \qquad B^I_{8m} = -B'_{9m}, \qquad B^I_{9m} = B'_{8m}, \qquad B^I_{89} = 0. \tag{3.30}$$

The set of backgrounds given in (3.28), (3.29) and (3.30) will be very useful to study the simple orbifold theory in the next section. The above form of metric can be recast in a more suggestive way which would clarify the topology of the six-manifold on the heterotic side. To see this let us first define a vector A as

$$A = \begin{pmatrix} dx^8 \\ dx^9 \end{pmatrix} . \tag{3.31}$$

Using this expression we see that metric of the six-dimensional manifold $K3 \times T^2/\mathbb{Z}_2$ on the type-IIB side takes the form

$$ds^2 = \Delta \tilde{g}_{mn} dx^m dx^n + \Delta A^\top g A \, . \tag{3.32}$$

To write the metric on the type-I side we use the set of equations (3.21), (3.23) and (3.25). The metric has a fibration structure as has been discussed earlier [3]. The novelty of the present case is that we are taking a non-trivial complex structure into account. This makes the details more interesting. The metric becomes

$$\begin{aligned} ds^2 \;&=\; \Delta \tilde{g}_{mn} dx^m dx^n + \frac{A^\top g A}{\Delta \det g} + \frac{\det(b\sigma_1 + g\sigma_3)\cdot dx}{\Delta \det g}\left[dx^8 + \mathrm{tr}(b\sigma_4)\cdot dx\right] + \\ &+ \frac{\det(b\sigma_2 + g\sigma_1)\cdot dx}{\Delta \det g}\left[dx^9 + \mathrm{tr}(b\sigma_3)\cdot dx\right] . \end{aligned} \tag{3.33}$$

Observe that the directions x^8 and x^9 are non-trivially fibred on the $K3$ base. The twist in the fiber is governed by the B fields. Let us consider now the simpler case given by (3.27), which is the case considered earlier in [3]. Under this assumption the metric simplifies to

$$ds^2 = \Delta \tilde{g}_{mn} dx^m dx^n + \frac{\Delta^{-1}}{\mathrm{tr}(g\sigma_1)}\left[dx^8 + \mathrm{tr}(b\sigma_1)\cdot dx\right]^2 + \frac{\Delta^{-1}}{\mathrm{tr}(g\sigma_3)}\left[dx^9 + \mathrm{tr}(b\sigma_3)\cdot dx\right]^2 . \tag{3.34}$$

Finally the type-I dilaton is related to the type-IIB dilaton by

$$\phi^I = \phi - \log \Delta - \frac{1}{2}\log(\det g)\, . \tag{3.35}$$

Since T-duality is a perturbative symmetry it produces a consistent background of type I. We will perform various checks in the later sections to see that this is indeed the case. The type-I background with torsion, as we shall see, has to satisfy many consistency conditions. In the next section we will take a simple orbifold model and predict the various fields. Also since T-duality preserves supersymmetry, this will be a supersymmetric background in four dimensions. As shown in section 3.1 the supergravity description is valid for both theories, type IIB and type I. Therefore we expect our heterotic/type-I background to satisfy the conditions found in [1, 2] and [9].

Under an S-duality transformation we go to the heterotic background. In our notation the heterotic string will have a coupling

$$g_{\mathrm{het}} = e^{\phi_{\mathrm{het}}} = e^{-\phi}\Delta\, \mathrm{Pf}(g)\, , \tag{3.36}$$

where by Pf we denote the pfaffian of g.

4. Torsional background: detailed analysis

In the earlier sections we described a framework to study a torsional background which can be *derived* from a consistent background of $\mathcal{M}$-theory. As we saw, a warped metric in $\mathcal{M}$-theory gives us, under a series of U-duality transformations, a type-I background with fluxes turned on. By performing an S-duality transformation we get a compactification of the heterotic string on a manifold with torsion. This background, as briefly alluded to in the introduction, has to satisfy a number of constraints. In this section we show that the type-I or heterotic background originating from the $\mathcal{M}$-theory compactification on $T^8/\mathcal{G}$ satisfies all the torsional equations.

4.1 $T^8/\mathcal{G}$ orbifold example in detail

We start with $\mathcal{M}/\mathcal{F}$-theory on $T^8/\mathcal{G}$, where $\mathcal{G}$ is the orbifolding group. This corresponds to the type-IIB theory compactified on the orientifold $T^2/\mathbb{Z}_2 \times T^4/\mathcal{I}_4$, where the $\mathbb{Z}_2$ action has been defined earlier.

We choose the G-flux of $\mathcal{M}$-theory as

$$\begin{aligned}\frac{G}{2\pi} &= A d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^3 \wedge dz^4 + B dz^1 \wedge d\bar{z}^2 \wedge dz^3 \wedge d\bar{z}^4 + \\ &\quad + C d\bar{z}^1 \wedge dz^2 \wedge dz^3 \wedge d\bar{z}^4 + D dz^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3 \wedge dz^4 + \sum_{i=1}^{4} F^i \wedge \Omega^i\,, \end{aligned} \tag{4.1}$$

where z^i are the complex coordinates of T^8 and Ω^i_{mn} are the harmonic 2-forms on $T^4/\mathcal{I}_4$. From the form of G we see that it has a constant part, which is spread over the full 4-fold, and a localized part which is concentrated at the singular regions. When we go to type IIB by shrinking the $\mathcal{M}$-theory two-torus (parametrized by z^4) to zero size, the constant part gives rise to the 3-form fields and the localized part appears as gauge fluxes on the 7-branes. This form of the G flux is *necessary* to get a consistent solution in the heterotic theory. As we shall discuss later, in the absence of the localized flux, there is probably no warped solution. In the next couple of sections we shall however suppress this contribution and will assume that the G-fluxes are constant for our case. This is not so unreasonable. The contribution from the localized fluxes come as $\mathcal{O}(\alpha')$ compared to the other terms in heterotic theory.

Therefore let us reduce the result (4.1) along the z^4 direction to go to the type-IIB theory. The constants A, B, C, D are related by the identity

$$\int_{\text{vol}} AB + CD = \frac{\chi}{24}\,, \tag{4.2}$$

where χ is the Euler characteristic of the 4-fold $T^8/\mathcal{G}$.

Demanding that G is real implies further constraints on A, B, C and D. We set $A = \bar{B}$ and $C = \bar{D}$ and we introduce the notation

$$A = \alpha_1 + i\alpha_2 \qquad \text{and} \qquad C = \beta_1 + i\beta_2\,. \tag{4.3}$$

The anomaly condition (4.2) in the presence of n M2-branes will constrain α_i, β_j by the following relation:

$$4(\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2) + n = 24\,. \tag{4.4}$$

To derive (4.4) we have assumed

$$\int_{\Gamma_x^k} dz^i = \delta_k^i \qquad \text{and} \qquad \int_{\Gamma_y^k} dz^i = i\delta_k^i\,, \tag{4.5}$$

as our periods for a smooth manifold with $\Gamma^i_{x,y}$ being the x, y one-cycles. Furthermore, we use the following quantization conditions:

$$\alpha_1 \pm \beta_1 \in 2\mathbb{Z} \qquad \text{and} \qquad \alpha_2 \pm \beta_2 \in 2\mathbb{Z}\,. \tag{4.6}$$

These quantization conditions take into account that $G/2\pi$ takes half-integer values when integrated over all 4-cycles. This also holds for the twisted sectors. This is a subtle issue and has to be dealt with case by case for any given orbifold. More details of this have appeared in [3]. Moreover, the quantization condition is modified from the usual cases because the orbifold action reduces the volume of the cycles. For the above case the volume of the 4-cycles is reduced by a quarter.

Let us consider $dz^4 = dx^{10} + \varphi dx^{11}$, where φ is the axion-dilaton combination which we have defined in section 3.2. This decomposition can be used to get the 3-forms on the base $K3 \times T^2/\mathbb{Z}_2$. We set

$$\frac{G}{2\pi} = H \wedge dx^{10} - H' \wedge dx^{11}\,. \tag{4.7}$$

Therefore after lifting to $\mathcal{F}$-theory (or type IIB) the H and H' fields are

$$\begin{aligned}
H \;&= A d\bar z^1 \wedge dz^2 \wedge d\bar z^3 + D dz^1 \wedge d\bar z^2 \wedge d\bar z^3 + \\
&\quad + B dz^1 \wedge d\bar z^2 \wedge dz^3 + C d\bar z^1 \wedge dz^2 \wedge dz^3\,, \\
H' \;&= \varphi(-A d\bar z^1 \wedge dz^2 \wedge d\bar z^3 - D dz^1 \wedge d\bar z^2 \wedge d\bar z^3) - \\
&\quad - \bar\varphi(B dz^1 \wedge d\bar z^2 \wedge dz^3 + C d\bar z^1 \wedge dz^2 \wedge dz^3)\,.
\end{aligned} \tag{4.8}$$

We observe that all the field components have one of their legs along the z^3 direction. Here z^3 is the complex coordinate of $T^2/\mathbb{Z}_2$. To make connection with the real coordinates described earlier, we use $z^3 = x^8 + ix^9$. In terms of complex coordinates the type-IIB metric is given by

$$ds^2 = 2(g_{1\bar 1} dz^1 d\bar z^1 + g_{2\bar 2} dz^2 d\bar z^2 + g_{3\bar 3} dz^3 d\bar z^3)\,. \tag{4.9}$$

Observe that we have used a diagonal complex structure of the tori with $\tau_{ij} = i\delta_{ij}$ for $i, j = 1, 2, 3$. To see whether this is the case we have to study the background a little more elaborately. Let us go back to real coordinates and define more generally

$$dz^i = dx^i + \tau^{ij} dy^j \qquad \text{with } \tau_{ij} = \text{diag}(\tau_1, \tau_2, \tau_3)\,. \tag{4.10}$$

The background can be determined in terms of a superpotential whose form is given by (2.1). For our case however it is[8]

$$W = (a_0 + \varphi c_0)\det\tau - \det\tau\,\mathrm{tr}[(a+\varphi c)^\top \tau^{-1}] - \mathrm{tr}[(b+\varphi d)^\top \tau] - (b_0 + \varphi d_0)\,. \qquad (4.11)$$

We will consider general diagonal matrices of a_{ij}, b_{ij}, c_{ij}, d_{ij} and generic c-numbers a_0, b_0, c_0, d_0. This is related to an example considered in [10, section 4.3]. To make contact with the notation of [10] we relabel our coordinates in the following way

$$(x_4, x_5, x_6, x_7, x_8, x_9) \to (x_1, y_1, x_2, y_2, x_3, y_3)\,. \qquad (4.12)$$

Therefore our background will be labelled by

$$\begin{aligned} a_{ij} &= \mathrm{diag}(a_1, a_2, a_3)\,, \qquad & b_{ij} &= \mathrm{diag}(b_1, b_2, b_3) \\ c_{ij} &= \mathrm{diag}(c_1, c_2, c_3)\,, \qquad & d_{ij} &= \mathrm{diag}(d_1, d_2, d_3)\,. \end{aligned} \qquad (4.13)$$

These matrices are used to write the background NS-NS and R-R 3-forms. We shall ignore the α' dependence of these forms for the time being. In the units of α' they are of order one and the localized fluxes in (4.1) are of order α'. The 3-forms are

$$\begin{aligned} H' &= a_0\alpha_0 + \mathrm{tr}(a^\top\alpha) + \mathrm{tr}(b^\top\beta) + b_0\beta_0\,, \\ H &= c_0\alpha_0 + \mathrm{tr}(c^\top\alpha) + \mathrm{tr}(d^\top\beta) + d_0\beta_0\,, \end{aligned} \qquad (4.14)$$

where α, β are defined in eq. (2.17) of [10]. The superpotential for our case can be written in terms of the above matrices as:

$$\begin{aligned} W = {} & (a_0 + \varphi c_0)\tau_1\tau_2\tau_3 - (a_1 + \varphi c_1)\tau_2\tau_3 - (a_2 + \varphi c_2)\tau_1\tau_3 - (a_3 + \varphi c_3)\tau_1\tau_2 - \\ & - (b_1 + \varphi d_1)\tau_1 - (b_2 + \varphi d_2)\tau_2 - (b_3 + \varphi d_3)\tau_3 - (b_0 + \varphi d_0)\,. \end{aligned} \qquad (4.15)$$

Let us now assume our background to satisfy

$$\tau_{ij} = \mathrm{diag}(i, i, i) \qquad \text{and} \qquad \varphi = i\,. \qquad (4.16)$$

Minimizing (4.15) with respect to φ, τ_{ij} we have the following set of ten relations between 16 variables:

$$\begin{aligned} a_1 + a_2 + a_3 &= b_0\,, & b_1 + b_2 + b_3 &= -a_0\,, \\ c_1 + c_2 + c_3 &= d_0\,, & c_2 + c_3 - b_1 &= a_0\,, \\ a_2 + a_3 + d_1 &= -c_0\,, & c_1 + c_2 - b_3 &= a_0\,, \\ a_1 + a_2 + d_3 &= -c_0\,, & d_1 + d_2 + d_3 &= -c_0\,, \\ a_1 + a_3 + d_2 &= -c_0\,, & c_1 + c_3 - b_2 &= a_0\,. \end{aligned} \qquad (4.17)$$

In the following we would like justify our choice of complex structure (4.16) by showing that it can be derived from the superpotential (4.15). Indeed, the 3-form (4.14), together

[8]Observe that this choice of superpotential differs from the usual choice by relative minus signs. This is because we preferred to choose $-H'$ as our RR three-from. This also implies that $G_3 = H' + \varphi H$ is a (2,1) form, as it should.

with the relations (4.17), reproduces the kind of background we have used in (4.8). There is a one-to-one correspondence between the parameters. The free parameters are given by

$$\begin{aligned} c_0 &= A+B+C+D\,, & d_0 &= i(-A-D+B+C)\,, \\ c_1 &= i(-A+D+B-C)\,, & d_1 &= -A+D-B+C\,. \end{aligned} \tag{4.18}$$

The rest of the parameters are determined in terms of c_0, d_0, c_1, d_1 by

$$\begin{aligned} c_2 &= b_1 = -b_2 = -c_1\,, \\ c_3 &= a_0 = -b_3 = d_0\,, \\ d_2 &= -a_1 = a_2 = -d_1\,, \\ d_3 &= a_3 = b_0 = -c_0\,. \end{aligned} \tag{4.19}$$

Including other terms in (4.1) we may also choose to cancel all the anomalies with branes. In general, the analysis of T-dualities would also apply in the presence of branes if we delocalize z^3 direction. In the presence of D3-branes we will obtain the type-I theory with 5-branes. These 5-branes will eventually be the heterotic 5-branes. We would like to consider a background for the heterotic theory without any small instantons. This would mean that the anomalies in type-IIB or $\mathcal{M}$-theory have to be cancelled by fluxes.

From the analysis done above we see that the only free parameters are c_0, d_0, c_1, d_1. The rest of the parameters are determined by some choice of these four variables. But we are not free to choose *any* variables. They are highly restricted by (4.6) and (4.4). It turns out that there is only *one* choice by which we can cancel anomalies and satisfy the quantization conditions

$$\begin{aligned} c_0 &= 4\,, & d_0 &= 0\,, \\ c_1 &= 4\,, & d_1 &= -4\,. \end{aligned} \tag{4.20}$$

This implies $A = 2+i$ and $C = i$. For the present case we do not require any wandering D3-branes to cancel the anomaly. However the above calculations did not take the other terms of (4.1) into account. The term with localized fluxes can take fractional coefficients, so that different values of $A, \ldots, D$ could be generated in our model.[9]

For the $\mathcal{M}$-theory lift of the model discussed in [10], i.e. for $\mathcal{M}$-theory compactified on the 4-fold $T^8/\mathcal{I}_8$, the quantization conditions and anomaly constraint are [3]

$$\alpha_1 \pm \beta_1 \in \mathbb{Z} \qquad \text{and} \qquad \alpha_2 \pm \beta_2 \in \mathbb{Z} \tag{4.21}$$

$$8(\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2) + n = 16\,. \tag{4.22}$$

For $A = 1+i$, $C = 0$ or $A = 1$, $C = 1$ we can have no wandering D3-branes in this model.

As discussed in the previous sections we also have four D7-branes and one O7-plane at every fixed point of the base $T^2/\mathcal{I}_2$. The metric $g_{3\bar{3}}$ should not be flat as we expect

[9] In fact this could actually be used to fix the moduli in [20]. There we needed a $D3-D7$ together and all the others to be far away. The axion behavior could now be chosen in such a way that we have non-zero value at every point. This non-zero value should be tuned in a way that most of the $D7$ and the (p,q) 7-branes are far away. A similar consideration apply to [19]. We choose an axion configuration which would tell us that *all* the 7 branes are far away.

backreactions from the branes and planes. In the general case, if we move one of the 7-branes in our system slightly then all the O-planes become (p,q) 7-branes under non-perturbative corrections. We shall not repeat the details of this mechanism here as this has been discussed extensively by Sen [18]. For our case (4.16) tells us that we have a charge cancelled situation $\tilde{\phi} = 0$ and therefore those moduli are fixed. Thus the metric will be flat everywhere except at the singular orbifold points. From the $\mathcal{M}$-theory point of view the X_8 term is responsible for the $D7$- and O7-planes. For this case

$$X_8 = \frac{3}{32} \sum_{z^i, w^j} \delta^4(z - z^i)\delta^4(w - w^j)\,, \tag{4.23}$$

where z^i, w^j are the fixed points of $T^4/\mathcal{I}_4 \times T^4/\mathcal{I}_4$.

From equation (4.8) it is easy to determine the B and B' backgrounds. They are:

$$\begin{aligned} B &= A\bar{z}^1 dz^2 \wedge d\bar{z}^3 + Bz^1 d\bar{z}^2 \wedge dz^3 - Cz^2 d\bar{z}^1 \wedge dz^3 - D\bar{z}^2 dz^1 \wedge d\bar{z}^3\,, \\ B' &= \frac{i}{g_B}\left(-A\bar{z}^1 dz^2 \wedge d\bar{z}^3 + Bz^1 d\bar{z}^2 \wedge dz^3 - Cz^2 d\bar{z}^1 \wedge dz^3 + D\bar{z}^2 dz^1 \wedge d\bar{z}^3\right). \end{aligned} \tag{4.24}$$

Observe that these B fields are not globally defined. To describe these fields we have to use patches on the six-manifold. H however is globally defined.

T-dualising to type I, the non zero fields are the metric $g^I_{a\bar{b}}$ and B^I fields. The metric takes the form

$$ds^2 = 2g^I_{a\bar{b}} dz^a d\bar{z}^b\,. \tag{4.25}$$

The type-I metric can be expressed in terms of the type-IIB metric and the NS-NS B field. Under T-duality the B field dissolves in the metric if it has components along the T-dual directions. From (4.24) we see that this is the case. Therefore (4.25) takes the form

$$\begin{aligned} ds^2 &= 2g_{1\bar{1}} dz^1 d\bar{z}^1 + 2g_{2\bar{2}} dz^2 d\bar{z}^2 + \\ &\quad + \frac{1}{2g_{3\bar{3}}}\Big(dz^3 + 2D\bar{z}^2 dz^1 - 2A\bar{z}^1 dz^2\Big)\Big(d\bar{z}^3 + 2Cz^2 d\bar{z}^1 - 2Bz^1 d\bar{z}^2\Big). \end{aligned} \tag{4.26}$$

As can be easily seen all other components of the metric are zero for the case that we are considering. The above metric (4.26) gets further simplified by imposing the condition that the G-fluxes should be real. It takes the following form

$$ds^2 = 2g_{1\bar{1}} dz^1 d\bar{z}^1 + 2g_{2\bar{2}} dz^2 d\bar{z}^2 + \frac{1}{2g_{3\bar{3}}}\left|dz^3 + 2D\bar{z}^2 dz^1 - 2A\bar{z}^1 dz^2\right|^2. \tag{4.27}$$

The B^I field of type I can be easily found from (3.30) using the set of fields (4.24) forming a vector bundle. It is given by

$$B^I = \frac{1}{g_B}\left(A\bar{z}^1 dz^2 \wedge d\bar{z}^3 + Bz^1 d\bar{z}^2 \wedge dz^3 - Cz^2 d\bar{z}^1 \wedge dz^3 - D\bar{z}^2 dz^1 \wedge d\bar{z}^3\right). \tag{4.28}$$

Indeed, from (4.24) one can easily see that $B_{m8} = -g_B B'_{m9}$ and $B_{m9} = g_B B'_{m8}$. This implies that the only non-vanishing components have one leg in the z^3 or $\bar{z}^3$ direction. Finally the type-I coupling is related to the type-IIB coupling via

$$g_I = \frac{g_B}{2g_{3\bar{3}}}\,. \tag{4.29}$$

From the metric components we see that type-I theory is compactified on a non trivial six-dimensional manifold. The geometry of the space is clear from (4.27). The z_3 and $\bar{z}_3$ directions are fibred in a specific way over the base. The twist of the fiber torus originates entirely from NS-NS B field of the type-IIB theory. This is a solution with torsion and therefore there are lots of constraints on it. In the next section we will discuss these constraints and show that our solution does indeed satisfy all the constraints.

4.2 Analysis of the torsional constraints

In the Einstein frame a type-I/heterotic background is supersymmetric when it satisfies the conditions[10]

$$\begin{aligned}\delta\psi_M &= \nabla_M\epsilon + \frac{1}{48}e^{-\phi/2}(\Gamma_M\mathcal{H} - 12\mathcal{H}_M)\epsilon = 0\,,\\ \delta\lambda &= \Gamma^N\nabla_N\phi\epsilon - \frac{1}{6}e^{-\phi/2}\mathcal{H}\epsilon = 0\,,\\ \delta\chi &= e^{-\phi/4}F_{MN}\Gamma^{MN}\epsilon = 0\,,\end{aligned} \tag{4.30}$$

for some Majorana-Weyl spinor ϵ. With indices M, N we label the ten-dimensional coordinates. Here χ is the gluino, ψ_K is the gravitino, λ is the dilatino and $\mathcal{H}$ is the 3-form field satisfying[11]

$$d\mathcal{H} = \frac{1}{2}\left[p_1(R) - p_1(F)\right] = \frac{1}{16\pi^2}\left(\mathrm{tr}\, R\wedge R - \frac{1}{30}\,\mathrm{Tr}\, F\wedge F\right). \tag{4.31}$$

We have also defined $\mathcal{H} \equiv \mathcal{H}_{MNP}\Gamma^{MNP}$ and $\mathcal{H}_M \equiv \mathcal{H}_{MNP}\Gamma^{NP}$.

The choice of coordinate system used in (4.30) is not appropriate because the metric is not the one that is used to measure distances. There is a more convenient coordinate system that is obtained by transforming the fields in the following way

$$g_{MN} \to e^{\phi/2}g_{MN}\,, \qquad \epsilon \to e^{\phi/8}\epsilon\,, \qquad \lambda \to e^{-\phi/8}\lambda\,, \qquad \psi_M \to e^{\phi/8}\left(\psi_M - \frac{1}{2}\Gamma_M\lambda\right). \tag{4.32}$$

We denote the resulting heterotic metric by $G_{m\bar{n}}$. It is related to the previous type-I metric by

$$G_{m\bar{n}} = \frac{2g_{3\bar{3}}}{g_B}\,g^I_{m\bar{n}}\,. \tag{4.33}$$

Here we have used that the heterotic coupling is related to the type-IIB coupling by

$$g_{\mathrm{het}} = e^{\phi} = \frac{2g_{3\bar{3}}}{g_B}\,. \tag{4.34}$$

For the particular case of compactification with a maximally symmetric space-time, unbroken supersymmetry implies that the warp factor is proportional to the coupling. It

[10] In this section $\phi \equiv \phi_{\mathrm{het}}$ unless mentioned otherwise.

[11] We differ here, for example, from [2] by a factor of 2. This factor is important to compare type-IIB and type-I theories. It is also important to determine the gravitational couplings on O-planes. For details see [22].

also requires the space-time to have vanishing cosmological constant [2]. To see this observe that for the choice of metric given in (4.32) the covariant derivative is shifted by

$$\nabla_M - \frac{1}{8}\Gamma_{MN}\partial^N \log\Delta\,. \tag{4.35}$$

This implies that the external component of the gravitino supersymmetry transformation law is given by

$$\delta\psi_\mu = \nabla_\mu\epsilon - \frac{1}{8}\Gamma_{\mu\nu}\partial^\nu(\log\Delta - \phi) = 0\,. \tag{4.36}$$

The above conclusions follow by contracting (4.36) with $\Gamma^{\mu\rho}\nabla_\rho$.

Our six-dimensional compact manifold is complex with fundamental form

$$J = iG_{a\bar{b}}\, dz^a \wedge d\bar{z}^b\,. \tag{4.37}$$

If $dJ = 0$ the manifold is Kähler and J is the Kähler form. The complex structure is constructed from the positive chirality spinor on the six-manifold, and is $\mathcal{H}$-covariantly constant

$$2\nabla_m J_n{}^p - \mathcal{H}_{qm}{}^p J_n{}^q - \mathcal{H}^q{}_{mn} J_q{}^p = 0\,. \tag{4.38}$$

Multiplying the above equation by the complex structure gives us the following important relation between the background 3-form and the fundamental form [1, 2, 9]:

$$\mathcal{H} = i(\bar{\partial} - \partial)J\,, \tag{4.39}$$

which can be expressed in terms of components as

$$\mathcal{H}_{\bar{m}\bar{n}p} = -2G_{p[\bar{m},\bar{n}]} \qquad \text{and} \qquad \mathcal{H}_{mn\bar{p}} = -2G_{\bar{p}[m,n]}\,. \tag{4.40}$$

Observe that the relation can now be written purely in terms of type-I variables. The type-I metric of course involves the warp factor in a complicated way as we have seen in earlier sections. The way we have derived this differs from [2] by a factor of 2.

By rescaling the fields as in (4.32) the set of equations in (4.30) becomes dilaton free. This implies that the killing spinor equation becomes

$$(\nabla\!\!\!\!/\, \phi - \frac{1}{6}\mathcal{H})\epsilon = 0\,, \tag{4.41}$$

which together with (4.40) gives rise to an equivalent relation between dilaton and $\mathcal{H}$-field

$$\mathcal{H}_{\bar{m}n\bar{p}}G^{n\bar{p}} = -\nabla_{\bar{m}}\phi\,. \tag{4.42}$$

In general the internal six-dimensional manifold will be non-Kähler. The shift from kählerity is given precisely in terms of the warp factor as

$$d^\dagger J = i(\partial - \bar{\partial})\log\Delta\,. \tag{4.43}$$

Therefore when $\Delta = 1$ or constant dilaton we have our usual Calabi-Yau compactification.

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In the type-IIB background most of the moduli fields of the internal manifold are fixed. From our earlier analysis (4.16) we see that $g_B = 1$. However, we will retain the coupling in all equations. Our final results will be independent of the choice of the coupling constant.

To verify the constraints we need the components of the heterotic metric. It is easy to extract from (4.26) and it is given by

$$G_{a\bar{b}} = \begin{pmatrix} G_{1\bar{1}} & G_{1\bar{2}} & G_{1\bar{3}} \\ G_{2\bar{1}} & G_{2\bar{2}} & G_{2\bar{3}} \\ G_{3\bar{1}} & G_{3\bar{2}} & G_{3\bar{3}} \end{pmatrix} = \frac{2}{g_B} \begin{pmatrix} g_{1\bar{1}}g_{3\bar{3}} + CDz^2\bar{z}^2 & -BDz^1\bar{z}^2 & D\bar{z}^2/2 \\ -AC\bar{z}^1 z^2 & g_{2\bar{2}}g_{3\bar{3}} + ABz^1\bar{z}^1 & -A\bar{z}^1/2 \\ Cz^2/2 & -Bz^1/2 & 1/4 \end{pmatrix}. \tag{4.44}$$

There are a set of 24 equations which follow from (4.40). In the following we will be demonstrating that these equations are consistent.

Using the duality relations derived in the previous sections we find the following equations

$$\mathcal{H}_{23\bar{1}} = \frac{C}{g_B} = -2G_{\bar{1}[2,3]} \qquad \text{and} \qquad \mathcal{H}_{13\bar{2}} = -\frac{B}{g_B} = -2G_{\bar{2}[1,3]}, \tag{4.45}$$

which are consistent. Observe that since the results are written in terms of the field strength, they are globally defined over the full six-manifold. The other non-vanishing components are

$$\mathcal{H}_{12\bar{1}} = 0 \qquad \text{and} \qquad G_{\bar{1}[1,2]} = \frac{1}{g_B}\frac{\partial}{\partial z^2}(g_{1\bar{1}}g_{3\bar{3}}) + \frac{CD}{g_B}\bar{z}^2 . \tag{4.46}$$

While the other component is

$$\mathcal{H}_{21\bar{2}} = 0 \qquad \text{and} \qquad G_{\bar{2}[1,2]} = -\frac{1}{g_B}\frac{\partial}{\partial z^1}(g_{2\bar{2}}g_{3\bar{3}}) - \frac{AB}{g_B}\bar{z}^1 , \tag{4.47}$$

and their complex conjugates.

If the identity (4.39) has to hold then we expect (4.46) and (4.47) to vanish. Therefore we expect

$$\frac{\partial}{\partial z^2}(g_{1\bar{1}}\ g_{3\bar{3}}) + CD\bar{z}^2 = \frac{\partial}{\partial z^1}(g_{2\bar{2}}\ g_{3\bar{3}}) + AB\bar{z}^1 = 0 . \tag{4.48}$$

The metric of the internal space in the type-IIB theory is basically the flat metric on a square torus T^6 multiplied by a warp factor Δ as shown in (3.19). After taking the warp factor dependence into account (4.48) become linear equations for the warp factor that take the form

$$\frac{\partial \Delta^2}{\partial z^2} + CD\bar{z}^2 = \frac{\partial \Delta^2}{\partial z^1} + AB\bar{z}^1 = 0 . \tag{4.49}$$

These linear equations have an origin in the type-IIB side of the duality. Indeed from the definition and self-duality of $\tilde{F}_5$ we obtain the equation

$$\star\, dD^+ = \frac{1}{2}(B \wedge H' - B' \wedge H) . \tag{4.50}$$

We have also seen that the condition for unbroken supersymmetry implies a relation between D^+ and the warp factor. This relation implies that (4.50) takes the form

$$\star\, d\Delta^{-2} = \frac{g_B}{2}(B \wedge H' - B' \wedge H) . \tag{4.51}$$

The Hodge $\star$-operator in (4.50) and (4.51) is defined with respect to the ten-dimensional metric. In order to compare with (4.49) we have to transform this to the Hodge operator with respect to the flat metric which we denote by $\tilde{\star}$. The result is

$$\tilde{\star}\, d\Delta^2 = i\left(AB\bar{z}_1 dz^1 \wedge dz^2 \wedge d\bar{z}^2 \wedge dz^3 \wedge d\bar{z}^3 + CD\bar{z}^2 dz^2 \wedge dz^1 \wedge d\bar{z}^1 \wedge dz^3 \wedge d\bar{z}^3 - \text{c.c.}\right). \tag{4.52}$$

After expanding in components we obtain exactly (4.49). This explains the type-IIB origin of the constraints (4.40). Note that (4.52) also implies that the warp factor is z_3 and $\bar{z}_3$ independent. This is consistent with our assumptions.

By acting with the adjoint of d on (4.52) it is easy to obtain the quadratic warp factor equation

$$\nabla^2\Delta^2 = -2(AB + CD) = -2(\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2)\,, \tag{4.53}$$

which from the $\mathcal{M}$-theory point of view takes the form

$$\nabla^2 e^{3\Phi/2} = -\frac{1}{2}\star\,(G \wedge G) = -2\,(\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2)\,. \tag{4.54}$$

We have used the fact that the volume of the fundamental domain is reduced by 1/4. The equations (4.53) and (4.54) are identical if

$$\Delta^2 = e^{3\Phi/2} \tag{4.55}$$

which is indeed the case because it is required by supersymmetry [5].

Observe that in the two equations above (4.53) and (4.54) we did not incorporate the delta function source terms. It is straightforward to incorporate these in (4.54) but it is not so easy to see how they appear in (4.53). This problem is partly related to the fact that we have taken $g_{3\bar{3}}$ to be a flat metric. Because of the presence of orbifold points the flat choice would actually not really work. To get the exact form of $g_{3\bar{3}}$ one has to solve Einstein's equation in the presence of fluxes. However, instead of doing this we can *still* view $g_{a\bar{b}}$ to be flat at all points and incorporate additional delta function contributions by hand. In other words, if we call the actual laplacian, which takes into account the singularities, $\Box$, then the relation between the two laplacians is

$$\Box = \nabla^2 + \mathrm{x}\,, \tag{4.56}$$

where x takes the singularites into account. From dimensional analysis we immediately see that x is of order L^{-2}, where L is the typical length scale on the six-manifold. The singularities should contribute delta functions to x, and thus the most general form for x is

$$\mathrm{x} = N \sum_{z^i, w^j} \frac{\delta^2(z - z^i)\ \delta^4(w - w^j)}{\Delta^2}\,, \tag{4.57}$$

where N is a number that could be determined from actually solving the gravity equations and taking the gravitational effects of D7-branes and O7-planes into account. Combining (4.57), (4.56) and (4.53) we get

$$\Box\Delta^2 = -2(\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2) + N \sum_{z^i, w^j} \delta^2(z - z^i)\delta^4(w - w^j)\,, \tag{4.58}$$

and using (4.23) we can confirm (4.55).

We should mention the following important point. In deriving the type-I background we have assumed that all the fields are independent of the $z^3, \bar{z}^3$ directions. However, (4.58) do not reflect this fact. It is of course true that the correct warp factor equation should not involve delocalization, but the T-duality rules we have used are not attuned to this. A more general T-duality rule is required to get the background that includes the delta function sources. But to the lowest order in α' we have obtained the correct answer and we have verified that it is consistent.

For the simpler case when we ignore the backreaction from D7-branes we can express the solution of (4.58) in terms of the Green's functions of the compact space K. Because of the compactness of the internal manifold there exists an IR cutoff. This IR cutoff appears as the volume factor in the Green's function [33]

$$\Box K(z, z^i, w, w^j) = \delta^2(z - z^i)\ \delta^4(w - w^j) - \frac{1}{\tilde{v}v}\,, \tag{4.59}$$

where $\tilde{v}$ and v have been defined in section 3.2. The backreaction from the D7-branes becomes important when the charges are not cancelled locally. The localized delta function source that has appeared in (4.58) originate from the X_8 term of $\mathcal{M}$-theory. As we discussed in (4.1), there is yet another localized source which is responsible for the gauge fields on the D7-branes. We have been neglecting that term but it also contributes to H and H' in (4.8). The contribution is such that

$$\tilde{\star}(H \wedge H') = 4(\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2) + \sum_{i=1}^{4} \tilde{\star}\,\mathrm{tr}(F^i \wedge F^i)\ \delta^2(z - z^i)\,. \tag{4.60}$$

We are working at the supergravity level in which we do not expect to see non-abelian gauge fields. But we have included a trace in front of the gauge fields in (4.60) because we will argue later that non-abelian gauge fields should appear after taking wrapped M2-branes into account. The non-abelian gauge symmetry we expect for our model is D_4^4. Indeed, we have four points on the base with one O7-plane and 4 D7-branes at every point. The D7-branes generate an $\mathrm{U}(4)^4$ gauge symmetry which is broken by the O7-planes to an $\mathrm{SO}(8)^4 = D_4^4$ symmetry.

For the orbifold case the curvature are localized at the fixed points. This is apparent from (4.58) and (4.23). Therefore the equation of warp factor will modify from (4.58) to

$$\Box\Delta^2 = -2(\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2) + \sum_{i,j} \delta^2(z - z^i)\left[N\ \delta^4(w - w^j) - \tilde{\star}\,\mathrm{tr}(F^j \wedge F^j)\right]. \tag{4.61}$$

Comparing with (4.59) we seem to be missing some volume factors. The reason for this is that we have taken unit volumes throughout.

Let us consider now a simpler situation where the instanton term is a delta function. The warp factor can actually be derived for this case following the calculations of [33]. It is given by

$$\Delta^2 = c_0 + c_1 K(z, z^i, w, w^j)\,, \tag{4.62}$$

where c_0, c_1 are constants and K is the same Green's function as in (4.59). In the limit in which the size of the six-manifold is very large i.e. when $|z|, |w| \to \infty$, we have

$$\Delta = \sqrt{c_0}\,, \qquad \Gamma^c_{\bar{b}c} = 0 \qquad \text{and} \qquad dJ = 0\,, \tag{4.63}$$

implying that the internal manifold is Ricci-flat and Kähler. Since we also have a vanishing first Chern-class, (4.63) would imply that at large radii these manifolds are Calabi-Yau 3-folds. For the case that the radius is a parameter of these string compactifications, quantum effects will prefer large radii and hence Calabi-Yau manifolds, as has been argued by Dine and Seiberg in [34]. This would be contradictory! Non-vanishing *total* fluxes like the ones we have in the present model can not be supported by Calabi-Yau 3-folds. Indeed, when the heterotic string is compactified on Calabi-Yau 3-folds fluxes are not allowed if we want to preserve some supersymmetry. Therefore we would find that these compactifications are inconsistent. This puzzle could be resolved if the solutions we constructed would contain a scale α' which could be used to actually fix the radius. Therefore these manifolds will not have a large radius limit. Some of these manifolds have also been studied recently in [35].

Up to now we have mostly ignored the localized fluxes appearing in (4.1). In the following we would like to study the condition for unbroken supersymmetry on those fluxes. The generic supersymmetry condition for G-fluxes is given in (1.1). For a choice of Kähler form J it is easy to see that the constant part of (4.1) satisfies (1.1). For the localized piece of G we find

$$G_{a\bar{b}c\bar{d}}J^{a\bar{b}} = i\sum_{i=1}^{4} g^{a\bar{b}}F^i_{a\bar{b}}(z^1,z^2,\bar{z}^1,\bar{z}^2)\Omega^i_{c\bar{d}}(z^3,z^4,\bar{z}^3,\bar{z}^4) = 0\,. \tag{4.64}$$

Here we have used that $\Omega \wedge J = 0$ because of the anti-self-duality of Ω. The form Ω is a $(1,1)$-form and not a $(2,0)$- or $(0,2)$-form because as discussed in (2.20) it is anti-self-dual. In fact, for $T^4/\mathcal{I}_4$ sixteen of these localized forms contribute to $h^{1,1}$ in addition to the already existing four $\mathcal{I}_4$ invariant forms $dz^i \wedge d\bar{z}^j$ to complete the twenty $(1,1)$-forms. Also since we are performing T-dualities the fields $F_{a\bar{b}}$ and $g_{a\bar{b}}$ are independent of the (z^3,z^4) directions. We do not need an explicit representation of Ω, but to extract information from (4.64) we need the orthogonality condition

$$\int_{K3}\Omega^i_{c\bar{d}}\Omega^j_{e\bar{f}} = (16\pi)^2\delta^{ij}\delta_{ce}\delta_{\bar{d}\bar{f}}\,, \tag{4.65}$$

where we have set the mass parameter on the right hand side of (4.65) equal to 1 (see for example [36]).

Integrating (4.64) over the 4-manifold $K3$ and using the relation (4.65) we obtain the condition for unbroken supersymmetry satisfied by the gauge fluxes

$$g^{a\bar{b}}F^i_{a\bar{b}} = 0\,. \tag{4.66}$$

This is of course the expected primitivity condition for the gauge fields on the D7-branes. This serves as another confirmation that the localized fluxes are the 7-brane gauge fields.

The above analysis only gave us the gauge fields that are in the Cartan subalgebra of D_4^4. The couplings and lagrangian of the ten-dimensional fields involved in the decomposition (4.1) and (4.7) can be worked out easily. By plugging in the value of G-flux (4.1) one can show that the lagrangian of $\mathcal{M}$-theory decomposes as

$$\int G\wedge\star G \to \int d^{10}x\,\left(g_B^{-2}|H|^2 + |H'|^2\right) + \sum_{i=1}^{4}\int d^8\sigma |F^i|^2$$

$$\int C \wedge G \wedge G \to \int D^+ \wedge H \wedge H' + \sum_{i=1}^{4} \int d^8\sigma D^+ \wedge F^i \wedge F^i \,. \tag{4.67}$$

The first terms appear in the type-IIB bulk and the second terms are interactions on the D7-brane world-volume. The $C \wedge X_8$ term gives rise to the couplings on the D7-branes and O7-planes only and it gives no contributions to the bulk interactions. We have used the orthogonality condition for the components of Ω to get the interactions of the D7-brane world-volume gauge fields. As we see, this analysis only gives the *abelian* part of the gauge group (i.e. the Cartan subalgebra). By taking wrapped M2-branes on the vanishing cycles at the fixed points into account we can argue for the complete non-abelian symmetry of our model. These M2-branes are wrapping 2-cycles of the 4-fold and are therefore different from the anomaly cancelling M2-branes in the $(2+1)$-dimensional space-time. Using U-duality it should be possible to see a similar kind of relation for the heterotic gauge fields. We would need a similar $(1,1)$-form for our six-manifolds. We postpone the details of this analysis to the future.

Finally, as discussed by Strominger, the six-manifold supports a unique holomorphic $(3,0)$-form. This form is in the middle dimensional cohomology. Let us recall the metric of the six-manifold again (we take $g_B = 1$ for simplicity here):

$$ds^2 = 4g_{3\bar{3}}g_{1\bar{1}}dz^1 d\bar{z}^1 + 4g_{3\bar{3}}g_{2\bar{2}}dz^2 d\bar{z}^2 + \left(dz^3 + a^i dz^i\right)\left(d\bar{z}^3 + b^j d\bar{z}^j\right), \tag{4.68}$$

where a_i, b_j are complex variables that can be obtained from (4.26). The form of the metric reminds us of the four-dimensional Taub-NUT space. The $(3,0)$-form can be constructed from the negative chirality spinors that are $\mathcal{H}$-covariantly constant

$$\omega = \frac{1}{2} e^{\alpha\phi} \eta_-^\top \Gamma_{123} \eta_- dz^1 \wedge dz^2 \wedge dz^3 \,, \tag{4.69}$$

where α is a constant. We have to determine this constant for our case since our conventions are different to those of [2], and therefore α will be different. We have an additional factor of $1/2$ in (4.69) because for our space dz^3 shifted according to

$$dz^3 \to \frac{1}{2} dz^3 + D\bar{z}^2 dz^1 - A\bar{z}^1 dz^2 \,. \tag{4.70}$$

This $(3,0)$-form should be independent of background fields. In particular we expect it to be independent of dilaton, i.e.

$$\partial_\phi \omega = 0 \,. \tag{4.71}$$

From the form of (4.69) it is not *a-priori* clear how this could be because the dilaton appears explicitly in the definition of ω. On the other hand, it is clear from the set of transformations (4.32) that the the dilaton dependence enters secretly after the rescaling

$$\Gamma_a \to e^{\frac{\phi}{4}} \Gamma_a \,, \qquad \text{and} \qquad \eta_- \to e^{-\frac{\phi}{8}} \eta_- \,. \tag{4.72}$$

Using this we conclude $\alpha = -2$. The $(3,0)$-form is determined completely in terms of the complex structure alone. Because of the compactness of the six-manifold any holomorphic 3-form we construct in this space would necessarily be a constant multiple of ω.

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Thus once we know the holomorphic form and the complex structure of the six-manifold we can in principle determine the background $\mathcal{H}$ field and the dilaton ϕ. However, so far we have not addressed the crucial question of the existence of this form. It turns out that if $\mathcal{H} \neq 0$ but[12] $d\mathcal{H} = 0$ then there exists no holomorphic form on the six-manifold [37]. For our case $d\mathcal{H} \neq 0$ and the non zero value comes from the additional term that we have in (4.1). As discussed earlier this additional term comes from the localized G-fluxes near the singularities. On the type-I side these localized fluxes will appear as gauge fields on the D9-branes which, when S-dualised will be the heterotic gauge fields.

5. Discussion

In this paper we have given an explicit compactification of the heterotic string on a six-dimensional manifold which is non-Kähler and has vanishing first Chern class. We have seen that the 3-form flux is non-vanishing. The metric of the six-dimensional manifold is determined in (4.26) and is shown to satisfy the torsional constraints that are imposed by the form of the supersymmetry transformations. This explicit formulation of the background should be useful to study other non-Kähler manifolds with torsion that can arise by removing some of the restrictions that we have imposed in section 4.2. The metric for the most generic background is derived in section 3.3. Showing that the general case also satisfies the torsional constraints is mathematically intense and we have not attempted to do it here. We are implicitly assuming that we can cancel all the anomalies by fluxes without using any wandering D3-branes. In the presence of D3-branes the torsional equations will themselves get modified by the backreactions from heterotic instantons. A more detailed analysis is therefore required to verify the torsional constraints for this case. But it is clear from duality arguments that this is a valid supergravity background.

There are other interesting questions and open problems that need to be addressed to complete the story:

- From (3.33) the topology of these manifolds is clear. They have a specific fibration structure. But important questions like the Euler characteristics or the value of the Hodge numbers have to be answered. Since the metric and the topology is known, these details should not be too hard to determine.

- In [10] a type-IIB background is determined by minimizing the superpotential (2.1). We also expect the heterotic background to follow from a superpotential. The torsional equations (4.40) and the Donaldson-Uhlenbeck-Yau equation for the gauge bundle should follow from an effective action in four dimensions. This matter is under investigation and more details will appear elsewhere [38].

- As briefly discussed in the text, the standard embedding of the spin connection into the gauge connection will lead to a trivial dilaton and a constant warp factor [37].

[12]This eliminates the familiar Calabi-Yau compactifications where $\mathcal{H} = d\mathcal{H} = 0$.

Therefore, having a non-constant warp factor will depend on the details of the hermitian structure of our six-manifold and the choice of gauge connection. This aspect can be used for building models that are phenomenologically attractive.

- Throughout the paper we have mostly concentrated on the $\mathcal{M}$-theory compactification on $K3 \times K3$. For the compactification on $T^8/\mathcal{I}_8$ there will be no D7-branes or O7-planes in the type-IIB lift of the $\mathcal{M}$-theory solution. Therefore, if we cancel the anomalies completely by fluxes[13] there will only be an abelian gauge symmetry in the four-dimensional space-time. It is an interesting question to trace what really happened to the SO(32) or $E_8 \times E_8$ gauge bundles on the heterotic side.

- Another important question concerns the size limit of our manifolds. We showed that for large radii these manifolds become Calabi-Yau and that this was a contradiction because the constant total fluxes could not be supported by Calabi-Yau manifolds. As suggested by [2] the 3-form equation $dH = \alpha'(p_1(R) - p_1(F))$ has a scale α' which can be used to fix the radius. This is presently under investigation [38].

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